UK Junior Mathematical Olympiad 2001 Solutions

A1 **2:50** Six hours, i.e. one quarter of a day, will have elapsed. One quarter of the 10 "hour" day is $2\frac{1}{2}$ "hours" or 2 "hours" 50 "minutes".

A2 **6** The division operation is performed before the addition. Therefore $\square + 4 = 10$.

A3 **15** The four marked angles are the interior angles of a quadrilateral. Hence $x = 360 - (105 + 115 + 125) = 15$.

A4 **72 cm^3** Let the length, breadth and height of the cuboid be l cm, b cm and h cm. Then lb, bh and hl have values 24, 18 and 12, though not necessarily in that order.

Therefore: $lb \times bh \times hl = 24 \times 18 \times 12$.

In cm^3, the volume of the cuboid $= lbh = \sqrt{24 \times 18 \times 12} = \sqrt{12 \times 12 \times 36} = 12 \times 6 = 72$.

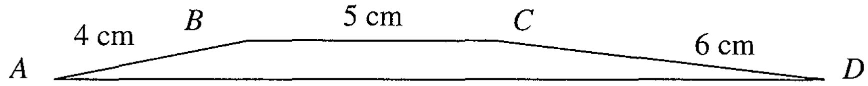
(Note that this method does not involve finding the dimensions of the cuboid. It is possible to do this: they are 6 cm, 4 cm and 3 cm.)

A5 **5** When written as a decimal, $\frac{3}{7} = 0.428571\,428571\,428571\ldots$.

The pattern repeats every six digits. Therefore the 96th digit after the decimal point is 1 and the 100th digit is 5.

A6 **$\frac{2}{9}$** Let the required fraction be x. Then $x + \frac{1}{2}x = \frac{1}{3}$, i.e. $\frac{3}{2}x = \frac{1}{3}$, i.e. $x = \frac{2}{9}$.

A7 **14cm** The length of AD must be less than 15 cm, since 15 cm would be its length if all four points lay in a straight line. However, by making angles ABC and BCD close to 180°, AD can be made close to 15 cm in length, as shown in the diagram.

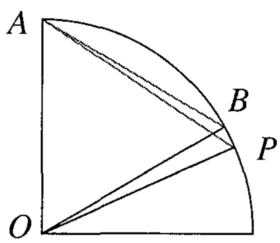


As the length of AD is a whole number of centimetres, its maximum value, therefore, is 14 cm.

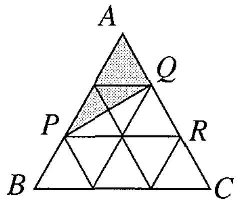
A8 **119** If the number is not prime and also not divisible by 2, 3 or 5, then it must have at least two factors which are prime numbers greater than 5. The required number is $7 \times 17 = 119$ as every whole number from 100 to 118 inclusive is either prime or divisible by 2, 3 or 5.

A9 **6°** Let O be the centre of the circle. Then $\angle AOB = \frac{360°}{6} = 60°$ and $\angle AOP = \frac{360°}{5} = 72°$. Therefore $\angle OAB = \frac{180° - 60°}{2} = 60°$, since $OA = OB$ and hence $\angle OAB = \angle OBA$. Similarly, $\angle OAP = \frac{180° - 72°}{2} = 54°$.

Hence $\angle BAP = 60° - 54° = 6°$.



A10 **$\frac{2}{9}$** Since all its sides are equal, triangle ABC is equilateral and the diagram shows that it may be divided up into 9 congruent equilateral triangles. The area of equilateral triangle APQ is half of the area of the triangle APR, which is four-ninths of the area of triangle ABC. Hence the area of triangle APQ is two-ninths of the area of triangle ABC.



(Note that the result will hold for any triangle in which the line PQ divides the sides in the ratio 1 : 2 as shown.)

B1 Let x, y and z be the numbers in the squares shown. Now the sum of the numbers from 1 to 7 inclusive is 28 and therefore the sum of the three equal totals will be $28 + x + 2$ since x and 2 both appear in two of the lines of three numbers. Thus $30 + x$ must be a multiple of 3 and hence x must also be a multiple of 3, i.e. $x = 3$ or 6.

1		
x	y	2
z		

If $x = 3$, the total of each line is $33 \div 3 = 11$ and therefore $y = 6$ and $z = 7$. The two remaining squares are filled with 4 and 5 so this may be done in two different ways.

If $x = 6$, the total of each line is $36 \div 3 = 12$ and therefore $y = 4$ and $z = 5$. The two remaining squares are filled with 3 and 7 so this may also be done in two different ways.

There are, therefore, four different ways of completing the grid:

1		4
3	6	2
7		5

1		5
3	6	2
7		4

1		3
6	4	2
5		7

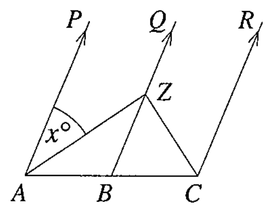
1		7
6	4	2
5		3

B2 (i) $2^2 + 5^2 = 29$; $2^2 + 9^2 = 85$; $8^2 + 5^2 = 89$; $8^2 + 9^2 = 145$.
Thus the first five terms are 25, 29, 85, 89, 145.

(ii) The full sequence is 25, 29, 85, 89, 145, 42, 20, 4, 16, 37, 58, 89, 145, 42, 20, 4, 16, 37, 58, 89, ...
Note that 89, 145, 42, 20, 4, 16, 37, 58 form a group of 8 terms which continue to repeat after the first three terms. Therefore the 4th, 12th, 20th, 28th, 36th, ..., 1996th, 2004th, ... terms are all 89. Hence the 2001st term is 16.

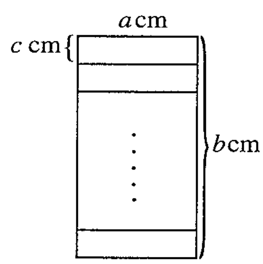
B3 (i) AZ bisects $\angle PAB$ and therefore $\angle ZAB = x°$. So $\angle PAB = 2x°$.
Therefore $\angle ZBC = 2x°$ (corresponding angles).

(ii) Since $\angle AZB = \angle ZAP$ (alternate angles), $\angle AZB = \angle ZAB$ so ΔABZ is isosceles. Thus $BZ = AB$. But $AB = BC$ so $BZ = BC$. This means that ΔZBC is also isosceles. So $\angle BCZ = \angle BZC = \angle RCZ$ (alternate angles) and hence $\angle BCZ = \angle RCZ$. Thus CZ bisects $\angle BCR$.



B4 Let the large rectangle have width a cm and height b cm. Let the smaller rectangles have height c cm (and they will have width a cm).

Then $2a + 2b = 300$ so $a + b = 150$ (1). Also $2a + 2c = 58$ so $a + c = 29$ (2). From (1) and (2), $b - c = 121$ (3). Also we know that c must divide into b and therefore, from (3), c divides into 121. So c must be 1 or 11 (since (2) shows $c = 121$ to be impossible).



Taking $c = 1$, from (2), we get $a = 28$ and then, from (1), $b = 122$. So the large rectangle is 28 cm by 122 cm.

Taking $c = 11$, from (2), we get $a = 18$ and then, from (1), $b = 132$. So the large rectangle is 18 cm by 132 cm.

B5 (i) The two-digit number whose digits are a and b has value $10a + b$.

We require that $10a + b = ab + a + b$ i.e. $9a = ab$ i.e. $a = 0$ or $b = 9$.

As a is the first digit of the number, it cannot be zero and hence a two-digit number has the required property if and only if its units digit is 9.

Apart from 49, therefore, the required numbers are 19, 29, 39, 59, 69, 79, 89 and 99.

(ii) We require that $100a + 10b + c = abc + a + b + c$ i.e. $99a + 9b = abc$ i.e. $abc - 99a = 9b$.

Therefore: $a(bc - 99) = 9b$. As b and c are both single digits, the maximum value of bc is $9 \times 9 = 81$, which means that the left-hand side of the equation is negative. This, however, is impossible and therefore we deduce that there are no three-digit numbers which are equal to the product of their digits plus the sum of their digits.

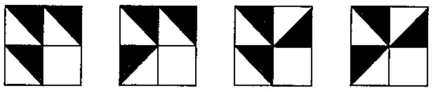
B6 (i) (a) If the tile in the top-left corner is in position A then the tile in the top-right corner is in position A or D and the tile in the bottom-left corner is in position A or C. These possible options are shown in the diagram. Once the positions of three tiles are fixed, the fourth one is determined, as shown in the bottom row. Thus there are four possible ways of covering the grid when the top-left tile is in position A.

A	A
A	

A	A
C	

A	D
A	

A	D
C	



A	A
A	A

A	A
C	C

A	D
A	D

A	D
C	B

(b) Two aspects of the process in part (a) should be highlighted. Firstly, when a tile is fixed, the tile to its right has two, but only two, possible positions; and the same is true of the tile below. Secondly, consider two squares which touch at a corner – they can be filled with tiles in any position – but, once they are filled, each of the two tiles which complete the 2×2 grid with them has one, and only one, possible solution.

(ii) We describe two different approaches here.

First, note that the top left corner can be filled in 4 different ways. Then there are 2 ways of filling each of the other places along the top row and down the first column. Thereafter, there is one and only one way of completing the grid. Thus the number of ways of filling the grid is $4 \times 2 \times 2 \times 2 \times 2 = 64$.

4	2	2
2	1	1
2	1	1

Alternatively, fix the tiles down the diagonal in any way at all – there are $4 \times 4 \times 4 = 64$ ways of doing this. Then there is exactly one way to complete the grid.

4	1	1
1	4	1
1	1	4

(iii) Extending the same arguments to an $n \times n$ grid, consider filling the top row and the first column, getting 4 choices for the top corner, 2 choices for each of the remaining $n - 1$ tiles along the top row and $n - 1$ tiles down the first column. Then the rest of the grid can be filled in exactly one way. That gives the total number of ways as $4 \times 2^{n-1} \times 2^{n-1} = 2^{2n}$.

Alternatively, consider filling the diagonal, from top left to bottom right, in an arbitrary fashion, having 4 choices for each tile, making 4^n choices in all. Then the rest of the grid can be filled in exactly one way. So the answer this way is 4^n. (Of course, this is the same number, written differently, since $4 = 2^2$ and so $4^n = 2^{2n}$.)